UNIVERSAL, CAVITY-SHAPE-INDEPENDENT RELATIONS FOR CAVITATION FLOW WITH SMALL CAVITATION NUMBERS

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The two-dimensional and axisymmetrical problem of cavitation flow of an ideal incompressible fluid past an arbitrary body in the presence of a source on the body or in the flow are studied. Universal, i.e., independent of the shape of the body, asymptotic (with respect to the cavitation number) relations between the resisting force, the length and width of the cavity, the cavitation number, and the intensity of the source are found.

It is proved in [1] that if, in the presence of a continuous flow past a body, a jet of fluid is ejected from the body in a direction oriented upstream, then a thrust appears and, for the case when the flow past the body occurs according to Kirchhoff's scheme, an example is given in which the resistance is two times lower than that of a body without injection but with the same asymptotic behavior of the cavity. It is shown in [2] that the replacement of the jet by a source gives a good approximation both for the force characteristics and for determining the form of the free streamlines. At the same time, modeling a jet flowing out of a body by a source makes it much easier to study the problem.

In [3], based on an exact solution of the two-dimensional problem of cavitation flow past a wedge with a source, the law governing the drop in the resistance and the appearance of a thrust is analyzed as a function of the intensity of the source and the angle of the wedge.

For axisymmetrical problems, the only theory which up to now has made it possible to obtain mathematically well-founded formulas for small cavitation numbers is the asymptotic theory of a thin body. According to this theory, the problem reduces, in the leading order approximation $|\ln \sigma| \gg 1$, to the solution of an ordinary differential equation. In the next approximation $\sigma \ll 1$, the equation turns out to be an integrodifferential equation [4, 5], whose solution has not yet been found.

In [6], numerical solutions for flow past cones based on Ryabushinskii's scheme are presented. Two universal relations, which are independent of the angle of the cone and which relate the resisting force, the cavitation number, and the elongation of the cavity, are presented and approximating formulas are given for them.

In this work, we formulate a variational principle for cavitation flow past bodies in the presence of hydrodynamic singularities. With the help of this principle, we find the specific form of the universal relations both for two-dimensional and for axisymmetrical flows with a source. In the two-dimensional case, we prove that these relations are asymptotically exact for arbitrary bodies.

1. Variational Principle. We shall examine the surface or contour $\partial\Omega$, over which an ideal incompressible fluid flows in a stationary manner. The velocity and pressure at infinity are equal to v_∞ and p_∞. The flow can contain point singularities, for example, sources. The problem is to find the general form of the functional U, whose variation for small changes in the boundary $\partial\Omega$ would be associated with the work performed by the pressure forces on the virtual displacement of the boundary by the relation

$$\delta U = \int_{\partial\Omega} (p - p_c)\, \delta n dS, \tag{1.1}$$

where δn is the displacement of the surface being varied $\partial\Omega$ along the outer normal to the body; p is the pressure of the liquid at the boundary $\partial\Omega$ which is determined from Bernoulli's integral; and p_c is the constant pressure.

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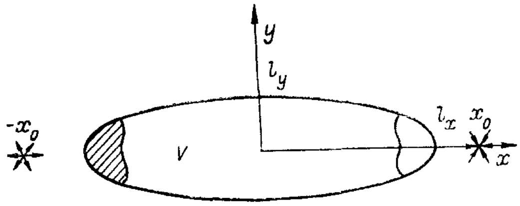

Fig. 1

In the absence of singularities, such a functional was found by Ryabushinskii [7] and is expressed in terms of the kinetic energy of the fluid. In the presence of singularities in the flow, the kinetic energy of the liquid is represented by a diverging integral. For this reason, regularization, which can be performed following [8, 9], is necessary. As a result, we obtain

$$U=\int\limits_V (p_0-p_c)\,d\tau-\int\limits_\Omega \frac{\rho}{2}(\mathbf{v}-\mathbf{v}_0)^2 d\tau, \tag{1.2}$$

where V is the region enclosed by the surface $\partial V = \partial\Omega$; Ω is the region outside of ∂V; $\mathbf{v}$ is the velocity of the fluid flowing over the surface ∂V in the presence of singularities; $\mathbf{v}_0$ and p_0 are the velocity and pressure of the potential flow with singularities in the absence of the surface ∂V; and p_c is the constant pressure on the boundary of the cavity.

Equations (1.1) and (1.2) permit giving the variational formulation of flow past a cavity based on Ryabushinskii's scheme in a flow with arbitrary singularities.

THEOREM. A cavitation flow, which is uniform at infinity, with arbitrary point singularities in the flow extremizes the quantity U in the sense that $\delta U = 0$ for arbitrary variations of the free surface on which the pressure of the fluid p_c is constant.

In the absence of singularities in the flow $\mathbf{v}_0 = \mathbf{v}_\infty$, $p_0 = p_\infty$, an analogous theorem was formulated by Ryabushinskii [7] and is the basis for the proof of the existence theorem for cavitation flows in both the two- and three-dimensional cases. These proofs can also be extended to cavitation flows with singularities in the flow.

We shall study a cavitation flow which is symmetrical relative to the x axis using Ryabushinskii's scheme. A source and a sink of intensity q are situated at the points $\mathbf{x} = -\mathbf{x}_0$ and $\mathbf{x} = \mathbf{x}_0$ (Fig. 1). For this case, the functional U can be represented in the following form (the derivation of this relation is given below):

$$U=-\frac{\rho}{2}v_\infty^2 M-2\rho v_\infty q\Phi(\mathbf{x}_0)-\rho q^2\Phi_0(\mathbf{x}_0,\mathbf{x}_0)+(p_\infty-p_\text{K})V, \qquad M=\int\limits_{\partial\Omega}\Phi\frac{\partial\Phi}{\partial n}dS, \tag{1.3}$$

where M is the virtual mass of the body V. The functions $\Phi(\mathbf{x})$ and $\Phi_0(\mathbf{x}_0, \mathbf{x})$, which are harmonic with respect to the variable x in the region Ω, approach zero at infinity, while at the boundary of the body $\partial\Omega$ they satisfy the conditions

$$\partial\Phi/\partial n = -\partial x/\partial n,\quad \partial\Phi_0/\partial n = -\partial\varphi_0/\partial n, \tag{1.4}$$

where φ_0 is the potential of the source and of the sink for the two-dimensional and axisymmetrical problems, respectively, having the following form:

$$\varphi_0=\frac{1}{2\pi}\ln\frac{r_+}{r_-},\quad \varphi_0=\frac{1}{4\pi}\left(\frac{1}{r_-}-\frac{1}{r_+}\right), \qquad r_+=[(x+x_0)^2+y^2]^{1/2},\ r_-=[(x-x_0)^2+y^2]^{1/2}. \tag{1.5}$$

It follows from the boundary conditions (1.4) that Φ is the potential of flow past the body V moving with a unit velocity, while Φ_0 is expressed in terms of the Green's function.

Derivation of Eq. (1.3). If the expression for p_0 from Bernoulli's integral is substituted into Eq. (1.2), we obtain

$$U = -\int_{\Omega} \frac{\rho}{2} (\mathbf{v} - \mathbf{v}_0)^2 d\tau - \int_{V} \frac{\rho}{2} v_0^2 d\tau - \left(p_c - p_\infty - \frac{\rho}{2} v_\infty^2\right) V. \tag{1.6}$$

The velocities $\mathbf{v}$ and $\mathbf{v}_0$ depend linearly on v_∞ and q and are represented in the form

$$\mathbf{v} - \mathbf{v}_0 = \mathrm{grad}(v_\infty \Phi(\mathbf{x}) + q\Phi_0(\mathbf{x}_0, \mathbf{x})), \ \mathbf{v}_0 = \mathrm{grad}(v_\infty x + q\varphi_0). \tag{1.7}$$

Substituting expression (1.7) into (1.6) and transforming to integrals over the boundary of the body ∂V using the Gauss–Ostrogradskii theorem, we obtain

$$U = -\frac{\rho}{2} v_\infty^2 M + (p_\infty - p_c) V - \rho v_\infty q I_1 - \rho q^2 I_2, \tag{1.8}$$

$$I_1 = -\frac{1}{2} \int_{\partial V} \left(\Phi \frac{\partial \Phi_0}{\partial n} + \Phi_0 \frac{\partial \Phi}{\partial n} - x \frac{\partial \varphi_0}{\partial n} - \varphi_0 \frac{\partial x}{\partial n}\right) dS,$$

$$I_2 = -\frac{1}{2} \int_{\partial V} \left(\Phi_0 \frac{\partial \Phi_0}{\partial n} - \varphi_0 \frac{\partial \varphi_0}{\partial n}\right) dS.$$

From Green's formula for harmonic functions and the boundary conditions (1.4), it follows that

$$\int_{\partial \Omega} \Phi \frac{\partial \Phi_0}{\partial n} dS = \int_{\partial \Omega} \Phi_0 \frac{\partial \Phi}{\partial n} dS, \ \int_{\partial V} x \frac{\partial \varphi_0}{\partial n} dS = \int_{\partial V} \varphi_0 \frac{\partial x}{\partial n} dS,$$

$$I_1 = -\int_{\partial V} \left(\Phi \frac{\partial \Phi_0}{\partial n} - \varphi_0 \frac{\partial x}{\partial n}\right) dS = \int_{\partial V} \left(\Phi \frac{\partial \varphi_0}{\partial n} - \varphi_0 \frac{\partial \Phi}{\partial n}\right) dS.$$

In the region Ω, the function Φ is harmonic, while φ_0 is harmonic everywhere except at the points $-x_0$ and x_0. Applying Green's theorem to a region, from which spheres with small radius R are removed, and passing to the limit $R \to 0$, we obtain

$$I_1 = 2\Phi(x_0). \tag{1.9}$$

Analogously, using the boundary conditions (1.4) and Green's theorem, we obtain

$$I_2 = \frac{1}{2} \int_{\partial V} \left(\Phi_0 \frac{\partial \varphi_0}{\partial n} - \varphi_0 \frac{\partial \Phi_0}{\partial n}\right) dS = \Phi_0(x_0, x_0). \tag{1.10}$$

Substituting Eqs. (1.9) and (1.10) into (1.8), we find (1.3), which is what had to be proved.

2. Variational Method for Solving the Problem of Cavitation Flow Past Body in the Presence of a Source. A variational method was proposed in [10] for calculating cavitation flows. For small cavitation numbers, this method permits obtaining asymptotically exact equations for both two-dimensional and axisymmetrical problems. This method can also be extended to cavitation flows with hydrodynamic singularities.

We shall examine a symmetrical cavitation flow with a source and a sink at the points $-x_0$, x_0 (see Fig. 1). For this scheme, the variational equation (1.1), where U is calculated according to Eq. (1.3), is applicable. Following [10], we introduce a two-parameter family of surfaces ∂V. For the parameters, we select l_x and l_y (half-length and half-width of the cavity, see Fig. 1). Then $U(l_x, l_y)$ is defined as a function of two parameters. In accordance with the variational equation (1.1), we obtain

$$F = \frac{1}{2} \frac{\partial U}{\partial l_x}, \ \frac{\partial U}{\partial l_y} = 0, \tag{2.1}$$

where F is the force exerted on the body by the liquid.

A force equal to the product of ρq by the velocity $\mathbf{v}'$, arising for all reasons other than the source, acts on the source; therefore

$$F_s = -\rho q v_\infty \left(\frac{\partial \Phi(x)}{\partial x} + 1\right) - \rho q^2 \left(\frac{\partial \Phi_0(x_0, x)}{\partial x} + v_s\right), \quad x = x_0, \tag{2.2}$$

where v_s is the velocity due to a source of unit intensity at a distance $2x_0$. Because of the symmetry of the function $\Phi_0(x_0, x)$, the partial derivatives with respect to the first and second arguments are equal at the point x_0.

From Eqs. (2.1) and (2.2) we obtain the total force acting on the body with a source $X = F + F_s$. In the limiting case when the source is located on the body $x_0 \to l_x$, we find

$$X = -\frac{\rho}{4} v_\infty^2 \frac{\partial M}{\partial l_x} - \rho v_\infty q \frac{\partial \Phi}{\partial l_x} + \frac{1}{2}(p_\infty - p_c)\frac{\partial V}{\partial l_x} - \frac{\rho q^2}{2}\left(\frac{\partial \Phi_0}{\partial l_x} + \frac{\partial \Phi_0(x_0, x_0)}{\partial x_0} + v_s\right). \tag{2.3}$$

We shall use the fact that the cavitation number $\sigma \ll 1$, and we shall estimate the order of magnitude of the terms entering into Eqs. (2.3). For estimates in the two-dimensional problem, we can choose the family of ellipses. Then

$$M = \pi l_y^2, \quad V = \pi l_x l_y, \quad \Phi(x_0) = \frac{l_y}{l_x - l_y}\left(x_0 - \sqrt{x_0^2 - l_x^2 + l_y^2}\right), \tag{2.4}$$

$$\Phi_0(x_0, x_0) \approx \frac{1}{2\pi}\left(\ln\frac{R}{x_0 - l_x} + \frac{3}{2}\frac{x_0 - l_x}{R} + O(x_0 - l_x)^2\right), \quad R = l_y^2/l_x,$$

where the expression for $\Phi_0(x_0, x_0)$ is taken from the known solution of the problem of a source near a circle of radius R. It follows from Eqs. (2.3) and (2.4) that in the limit $x_0 \to l_x$

$$c_* = \frac{X}{\frac{1}{2}\rho v_\infty^2 \pi l_y} = \frac{1}{2}\sigma - 2Q, \quad Q = \frac{q}{\pi v_\infty l_y}; \tag{2.5}$$

$$\sigma = 2\chi(1 + Q)^2, \quad \sigma = \frac{v_c^2 - v_\infty^2}{v_\infty^2}, \quad \chi = \frac{l_y}{l_x}. \tag{2.6}$$

From the exact solution of the cavitation problem, given below, we shall find that the quantity Q is limited from above $Q \le K\sigma$, where the constant K depends only on the shape of the body (for plates, for example, the exact lower limit is equal to $K = 1/6$). Therefore, in the leading order approximation $\sigma \ll 1$, we have $\sigma = 2X$.

From here it also follows that in calculating the total force and the degree of elongation of the cavity according to Eqs. (2.1)-(2.3), the term quadratic in q can be dropped in the functional U:

$$U_0 = -\frac{\rho}{2} v_\infty^2 M + (p_\infty - p_c)V - 2\rho v_\infty q \Phi(x_0), \tag{2.7}$$

$$X = \frac{1}{2}\frac{\partial U_0}{\partial l_x}, \quad \frac{\partial U_0}{\partial l_y} = 0.$$

It will be proved below that for the family of ellipses in the two-dimensional problem Eqs. (2.7) are asymptotically exact with respect to the cavitation number and do not depend on the form of the body.

It is useful to apply Eqs. (2.7) to the calculation of axisymmetrical cavitation flows, using the family of ellipsoids.

The virtual mass M of the ellipsoid and the potential of flow Φ past it are known to satisfy the following equations [11]:

$$M = Vm(\chi), \quad \Phi = l_x f\left(\frac{x_0}{l_x}, \chi\right), \quad m = -1 + (1 - \chi^{-2})(A(\chi) - \chi^{-2})^{-1}, \tag{2.8}$$

$$f(\xi, \chi) = (A(\chi) - \chi^{-2})^{-1}\left(\frac{\xi}{2(1 - \chi^2)^{1/2}}\ln\frac{\xi + (1 - \chi^2)^{1/2}}{\xi - (1 - \chi^2)^{1/2}} - 1\right),$$

$$A(\chi) = (1 - \chi^2)^{-1/2}\ln\left(\chi^{-1} + \sqrt{\chi^{-2} - 1}\right), \quad V = \frac{4\pi}{3} l_x l_y^2, \quad \chi = l_y/l_x.$$

When the source is located on the surface of the body $x_0 = l_x$, the following relations are valid:

$$l_x \partial M/\partial l_x = V(m - \chi m'),\ l_y \partial M/\partial l_y = V(2m + \chi m'),$$
$$f(1, \chi) = -m(\chi),\ \partial\Phi/\partial l_y = -m',\ \partial\Phi/\partial l_x = -1 - m + \chi m',$$
$$m' = dm/d\chi. \tag{2.9}$$

Substituting Eqs. (2.8) into (2.7), with the help of Eqs. (2.9) we obtain

$$c_* = \frac{X}{\frac{1}{2}\rho v_\infty^2 \pi l_y^2} = \chi m' - 2Q(1 - m + \chi m'),$$
$$\sigma = m + \frac{1}{2}\chi m' - \frac{3}{2}Q\chi m', \quad Q = \frac{q}{\pi\rho v_\infty l_y^2}. \tag{2.10}$$

For Q = 0 these equations coincide with the equations found previously in [10] using the same method and they approximate well the numerical solutions.

It is easy to obtain the asymptotic form of the relations (2.10) with a small cavitation number with the help of Eqs. (2.8):

$$c_* = 2\chi^2\left(\ln\frac{2}{\chi} - \frac{3}{2}\right) - 2Q, \quad \sigma = 2\chi^2\left(\ln\frac{2}{\chi} - \frac{5}{4}\right). \tag{2.11}$$

For Q = 0, $\sigma \ll 1$, in leading order the asymptotic form of these formulas agrees with Garabedian's asymptotic form [7].

3. Derivation of Asymptotic Formulas from the Exact Solution. We shall prove that Eqs. (2.6), obtained from the variational principle, are asymptotically exact. For this, we write out the system of equations (found by a well-known method [12]), giving the complete solution of the problem of cavitation flow past an arbitrary symmetrical contour with a source located on it according to Efros's scheme. Figures 2 and 3 show the plane of the flow z and the auxiliary plane ζ. The derivatives of the complex potential W with respect to the variables z and ζ have the form

$$\frac{1}{v_\text{к}}\frac{dW}{dz} = \frac{(\zeta + i)(\zeta - ih)(\zeta h - i)(\zeta - ik)(\zeta k - i)\,e^{F(\zeta)}}{(\zeta - i)(\zeta + ih)(\zeta h + i)(\zeta + ik)(\zeta k + i)},$$
$$F(\zeta) = iA_1\zeta - \frac{1}{3}iA_3\zeta^3 - \ldots,$$
$$\frac{dW}{d\zeta} = Nv_\text{к}\frac{(\zeta^2 - 1)(\zeta^2 + h^2)(h^2\zeta^2 + 1)(\zeta^2 + k^2)(k^2\zeta^2 + 1)}{\zeta(\zeta^2 + c^2)^2(c^2\zeta^2 + 1)^2(\zeta^2 + 1)}, \tag{3.1}$$

where ik and ih are the coordinates of the images of critical points inside the flow; ic is the coordinate of the image of a point at infinity in the flow in the complex ζ plane (see Fig. 3). For k = 1, Eqs. (3.1) coincide with the well-known equations [12] for flow past a curvilinear arc according to Efros's scheme.

The coefficients A_n determine the form of the contour. To Eqs. (3.1) we must add also Will's condition

$$\operatorname{Re}\left[i\frac{d}{d\zeta}\ln\frac{dW}{v_c dz}\right] = 0, \quad \zeta = \pm 1,$$

which is necessary so that the curvature of the free surface at the points A and B be finite, as well as the condition

$$\operatorname{res}\left[\frac{dz}{d\zeta};\quad \zeta = ic\right] = 0, \tag{3.2}$$

which indicates that the function $z(\zeta)$ after circumscribing the point z = ic returns to its initial value.

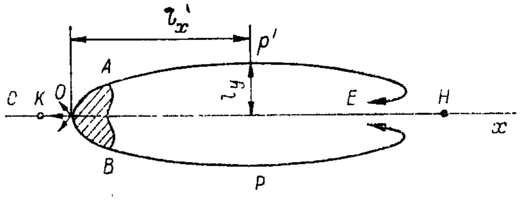

Fig. 2

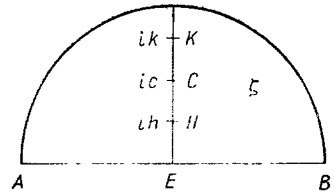

Fig. 3

The intensity of the source q and the flow rate in the stream δ are determined from the equations

$$q = \pi \operatorname{Re}\left(\operatorname{res}\left[\frac{dW}{d\zeta};\ \ \zeta = i\right]\right) = N v_{\mathrm{e}} \pi \frac{(h^2-1)^2(k^2-1)^2}{(c^2-1)^4}, \tag{3.3}$$

$$\delta = -\pi \operatorname{res}\left[\frac{dW}{d\zeta};\ \ \zeta = 0\right] = N v_{\mathrm{e}} \pi \frac{h^2 k^2}{c^4},$$

and the cavitation number σ is related to the parameters of the problem c, h, k, A_n by the relation

$$\left|\frac{dW(ic)}{v_{\mathrm{e}}\, dz}\right| = (\sigma+1)^{-1/2} = \left|\frac{(c+1)(c-h)(ch-1)(c-k)(ck-1)\mathrm{e}^{-A_1 c - \frac{A_3}{3}c^3}}{(c-1)(c+h)(ch+1)(c+k)(ck+1)}\right|. \tag{3.4}$$

The limit $\sigma \to 0$ corresponds to Kirchhoff's scheme and the points H and E approach, in this case, the point C at infinity. In the ζ plane this corresponds to the fact that the numbers c and h approach zero. It can therefore be assumed that for $\sigma \ll 1$, the parameter $c \ll 1$. We choose c to be an independent small parameter and we shall seek the dependences h(c) and σ(c) in the form of expansions with respect to this parameter

$$h = h_1 c + h_2 c^2 + \ldots,\ \sigma = \sigma_1 c + \sigma_2 c^2 + \ldots \tag{3.5}$$

Using Eqs. (3.1), we obtain

$$\frac{dz}{d\zeta} = \frac{N(\zeta^2-1)(\zeta+ih)^2(\zeta h+i)^2(\zeta+ik)^2(\zeta k+i)^2}{\zeta(\zeta+ic)^2(\zeta-ic)^2(\zeta+i)^2(c^2\zeta^2+1)^2\mathrm{e}^F}. \tag{3.6}$$

At the point ζ = ic, the function dz/dζ has a second-order pole, and the residue at this point is equal to the derivative of the factor multiplying $(\zeta - ic)^{-2}$. From here, the condition (3.2) assumes the form

$$\frac{2\zeta}{\zeta^2-1} + \frac{2}{\zeta+ih} + \frac{2h}{h\zeta+i} + \frac{2}{\zeta+ik} + \frac{2k}{\zeta k+i} - F'(\zeta) -$$

$$-\frac{1}{\zeta} - \frac{2}{\zeta+ic} - \frac{4c^2\zeta}{c^2\zeta^2+1} - \frac{2}{\zeta+i} = 0,\quad \zeta = ic.$$

Substituting into this equation the first expansion in (3.5) and equating the coefficients of c^{-1} and c^0, we obtain

$$h_1 = 0,\ h_2 = a = k + 1/k - 1 + A_1/2. \tag{3.7}$$

Substituting the expansion found for h(c) into (3.4) and taking into account terms of order c, we obtain the expansion of σ(c):

$$1 - (1/2)\sigma + \ldots = 1 + (-A_1 + 2 - 2a - 2/k + 2k)c + \ldots,\ \sigma = 8ac. \tag{3.8}$$

It is also interesting to calculate the length $2l_x$ and the width $2l_y$ of the cavity. For this, we define a point P in the free jet at which the velocity of the liquid is parallel to the x axis. We shall call the distance from the point P to the plate along the horizontal l_x

and the distance of this point from the axis of symmetry l_y (see Fig. 2). The coordinates of the point P in the ζ plane can be determined from the condition Im $dW/dz = 0$. In the complex ζ plane, this point is determined by the real number t, which approaches zero as $\sigma \to 0$. For this reason, we seek t in the form of the expansion $t = t_1 c + t_2 c^2 + \ldots$.

Retaining the leading terms in the expansion in powers of c in Eq. (3.1), we find

$$\frac{1}{v_c}\frac{dW}{dz} = 1 + \left(2\frac{t_1}{i} - 2\frac{ih_2}{t_1} - 2\frac{t_1}{ik} - 2\frac{t_1 k}{i} + iA_1 t_1\right)c.$$

From the condition that the imaginary part of this expression is equal to zero, we obtain

$$t = c + \ldots \tag{3.9}$$

The half-length l_x and the half-width l_y of the cavity for small σ can be determined from the equation

$$l_x - il_y = \int\limits_1^c \frac{dz}{dt}\,dt, \tag{3.10}$$

where the integral is the integral along the real axis of the function (3.6), which can be represented in the form

$$\frac{dz}{dt} = \frac{N\Phi(t, c)}{t(t^2 + c^2)^2}, \tag{3.11}$$

$$\Phi = \frac{(t^2 - 1)(t + ih)^2 (th + i)^2 (t + ik)^2 (tk + i)}{(t + i)^2 (t^2 c^2 + 1)^2 e^{F(t)}},$$

$$\Phi = \Phi_0 + \Phi_1 t + \Phi_2 t^2 + \Phi_3 t^3 + r(c, t)t^4,$$

$$\Phi_0 = h^2 k^2,\ \Phi_1 = -2k^2 hi + \ldots,\ \Phi_2 = -k^2 + \ldots,\ \Phi_3 = 2ik^2 a + \ldots,$$

where the dots indicate higher-order infinitesimals. The coefficient of the remainder r(c, t) is a continuous function in the closed region $0 \leqslant t \leqslant 1$, $0 \leqslant c \leqslant c_0$ and is therefore bounded.

Substituting the expansion (3.11) into (3.10), we obtain

$$l_x = N\int\limits_1^c \frac{(h^2k^2 - k^2t^2)^2\,dt}{t(t^2 + c^2)^2} \approx \frac{Nk^2}{4c^2}, \tag{3.12}$$

$$l_y = N\int\limits_1^c \frac{(2k^2ht - 2k^2at^3)\,dt}{t(t^2 + c^2)^2} \approx \frac{Nk^2 a}{c},\quad \chi = \frac{l_y}{l_x} = 4ac.$$

The asymptotic equation for the total resisting force can be obtained from the general equation [13], found from the law of the change of momentum

$$\frac{X}{(1/2)\rho v_\infty^2} = \frac{2(\delta - q)}{v_\infty} + \frac{2\delta v_c}{v_\infty^2}. \tag{3.13}$$

For the flow from the source q and the flow of the return stream δ, we obtain from Eqs. (3.3) and (3.12) up to leading order terms

$$Q = \frac{q}{\pi v_\infty l_y} = \frac{(k^2 - 1)^2 c}{k^2 a},\quad \frac{\delta}{\pi v_\infty l_y} = ac. \tag{3.14}$$

From here we obtain the equation for the force (2.6).

4. Analysis of Asymptotic Equations. We choose as the determining parameters the velocity at infinity v_∞, the flow rate of the source q which models the jet oriented along the flow, the cavitation number σ, and the maximum midsection of the cavity l_y. The remaining quantities (the resisting force X, the length of the cavity l_x, and the flow rate of Efros's return stream δ) can be calculated from these data. Introducing the dimensionless quantities, referred to the maximum midsection,

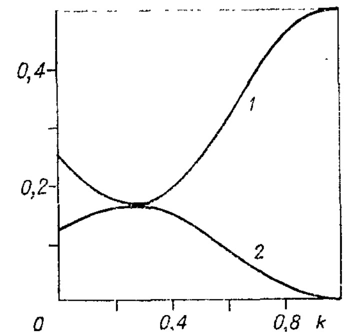

Fig. 4

$$c_* = \frac{X}{\frac{\pi}{2}\rho v_\infty^2 l_y}, \quad Q = \frac{q}{\pi v_\infty l_y}, \quad \bar{\delta} = \frac{\delta}{\pi v_\infty l_y}, \quad \chi = \frac{l_y}{l_x} \tag{4.1}$$

and expressing the parameter c in terms of σ according to Eqs. (3.8), we obtain the asymptotic equations (3.12)-(3.14) in the following compact form:

$$c_* = \frac{\sigma}{2} - 2Q, \quad \chi = \frac{\sigma}{2}, \quad \bar{\delta} = \frac{\sigma}{8}, \quad Q = \frac{(k^2-1)^2\sigma}{(8k^2a^2)}, \quad a = k + \frac{1}{k} - 1 + \frac{A_1}{2} \neq 0. \tag{4.2}$$

The first three expressions are universal for cavities of arbitrary form (with the exception of the degenerate case $a = 0$). An important property of the cavity is the fact that the degree of elongation of the cavity $1/\chi$ does not depend on the intensity of the source.

The form of the cavity affects only the change in the quantity Q. When the quantity Q reaches its maximum value Q_{max}, the resistance coefficient c_* will be minimum. An analysis of the equation for Q (4.2) shows that, for cavities with the parameter $-2 < A_1 \leqslant 2$ (including for the plate $A_1 = 0$), the largest value of the source is determined by the expression $Q_{max} = (\sigma/8)(1 - (2 - A_1)^2/16)^{-1}$. For cavities with the parameter $A_1 > 2$, we have $Q_{max} = \sigma/8$, $k = 0$. For the last case $A_1 \leqslant -2$, the quantity a vanishes for some value $0 \leqslant k \leqslant 1$, the cavities are compressed into the midsection, and the resisting force becomes a thrust.

The parameter k determines the position of the critical point in front of the body at which the velocity vanishes. For $k = 1$, when there is no source, the critical point is located on the body $Q = 0$, $c_* = \sigma/2$. As k is decreased, the critical point moves away from the body and in the limit $k \to 0$ it moves off to infinity. In this limiting case, $Q = \sigma/8$, $c_* = \sigma/4$, i.e., the resistance drops by a factor of two. This example corresponds to the case noted in [1].

From Eqs. (4.2), we can obtain an asymptotic law for the expansion of the cavity for $\sigma = 0$, if we write the equation for the resisting force in the dimensionless form

$$X = \frac{\pi}{2}\rho v_\infty^2 \frac{l_y^2}{l_x} - \rho v_\infty q. \tag{4.3}$$

For $q = 0$, we obtain the parabolic law for the expansion of the cavity found by S. A. Chaplygin (in the limit $\sigma \to 0$, the elliptical cavity transforms into a parabolic cavity with the parameter of the parabola l_y^2/l_x). The general form of Eq. (4.3) is obtained in [13].

It should be noted that Chaplygin's law is not universal. Bodies with a special form, for which $A_1 = -2$ and for which the asymptotic behavior of the free streamlines is entirely different (they converge at infinity), exist. Without the source, the resistance of such bodies is equal to zero, while in the presence of a source a thrust force acts on the body. The form of bodies in the degenerate case is easy to construct, starting from the exact equation (3.6), by selecting A_1 from the equation $a = 0$ and the remaining coefficients arbitrarily.

Curves 1 and 2 in Fig. 4 show the dependence of c_*/σ, Q/σ on k for a plate ($A_1 = 0$). For $k = 0$, the intensity of the source $Q = \sigma/8$ and the resistance coefficient is equal to $\sigma/4$, i.e., a factor of two smaller than with $Q = 0$. In the limit $\sigma \to 0$, this corresponds to the case noted in [1].

As is evident from Fig. 4, the lowest resistance for a fixed midsection $c_* = \sigma/6$ is obtained with $Q = \sigma/6$.

It is interesting to note that for $Q > \sigma/8$, one and the same value of the source intensity corresponds to two flow regimes. In addition, for all permissible source intensities, the quantity χ, up to small orders of σ, does not change.

In the absence of a source, from Eqs. (4.2) we obtain the results $A_1 \neq -2$, $c_* = \sigma/2$, $\chi = \sigma/2$, $\bar{\delta} = \sigma/8$, referring to flow past arbitrary contours according to Efros's classical scheme.

We have proved that the form of the relations (2.11) is analogous to the corresponding relations in the two-dimensional problem. In both cases, the resistance coefficient, referred to the midsection, decreases as Q increases and the degree of elongation of the cavity $1/\chi$ does not depend on the source intensity q. An important feature of the axisymmetrical problem is that for practical calculations, together with the asymptotically leading term of order $\chi^2 \ln \chi$, the term of order χ^2 must also be taken into account, even for very small cavitation numbers.

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